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LETTER TO THE EDITOR

Multisoliton solutions of nonlinear dispersive wave equations not soluble by the inverse method

R K Dodd, R K Bullough and S Duckworth

Department of Mathematics, University of Manchester Institute of Science and Technology, Manchester M60 1QD, UK

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Abstract. Numerical solutions of certain physically important generalized sine–Gordon equations display complex multisoliton behaviour. Two theorems show that these equations have neither Bäcklund transformations nor an infinity of polynomial conserved densities. Yet all known equations with multisoliton solutions obtained by the inverse method seem to yield both.

The 'double sine–Gordon (SG) equation'

$$\sigma_{xx}-\sigma_{tt} = \sin \sigma+\tfrac{1}{2}\sin \tfrac{1}{2}\sigma \tag{1}$$

occurs in the theory of the propagation of resonant ultra-short ($\lesssim 10^{-9}$ s) optical pulses through degenerate media (Lamb 1971, Duckworth *et al* 1975, to be called I). The atoms or molecules of the media are supposed to have a single significant frequency at resonance but the resonant transitions take place between pairs of $2J+1$ degenerate levels with selection rules $\Delta J = 0$, $\Delta M_J = 0$. The equation (1) applies specifically to the case $J = 2$. For arbitrary J equation (1) is replaced by the 'multiple sine–Gordon equation'

$$\sigma_{xx}-\sigma_{tt} = \sum_{m=1}^{J}\frac{m}{J}\sin\frac{m}{J}\sigma. \tag{2}$$

When $J = 1$ equation (2) becomes the simple (single) sine–Gordon equation $\sigma_{xx}-\sigma_{tt} = \sin\sigma$. This equation governs the propagation of sharp-line resonant pulses through non-degenerate media (Lamb 1971) and the phenomenon of self-induced transparency (SIT). It also has application to many other different physical problems. A multisoliton solution of this simple SG is known (Ablowitz *et al* 1973a, Caudrey *et al* 1973b, Hirota 1972). These analytical solutions are 'kink' solutions whose derivatives are pulse solutions: in the optical case these pulses are pulses of electrical field envelope. The pulse solutions break up into a series of hyperbolic secant pulses (single solitons) of characteristic 'areas' (cf eg Lamb 1971) each equal to 2π. These solitons have different amplitudes and hence different speeds $V < c$: V increases with pulse amplitude. Slusher and Gibbs (1972) report evidence of pulse break-up of this character in the SIT of non-degenerate ^{87}Rb vapour; Salamo *et al* (1974) report similar evidence in the SIT of degenerate and non-degenerate Na vapour. Other nonlinear dispersive wave systems show similar break-up and are known to have multisoliton solutions. For example, the Korteweg–de Vries equation governs such break-up in ion-acoustic waves in plasmas (Ikezi *et al* 1970) and Hirota (1971) has found a multisoliton solution of it.

A multisoliton solution of equation (2) has not been found for $J > 1$; but we have obtained (I) numerical solutions for $J = 2$ and $J = 3$ which show all the characteristic features of solitons. For example, when $J = 2$, so that (1) applies, the initial condition $\sigma = 0$ corresponds to an initially unexcited dielectric of five-fold degenerate two-level atoms. Single distortionless soliton-like pulses are double peaked with area 4π; break-up occurs into trains of such 4π pulses of different amplitudes whilst two 4π pulses can collide and pass through each other without change of shape and velocity and with phase shifts which are certainly small.

We have also observed features not seen previously in collisions between single solitons. In particular we report here a new type of soliton-like break-up and collision observed in numerical solutions of (2) in the case $J = 3$. Figure 1 illustrates one form of

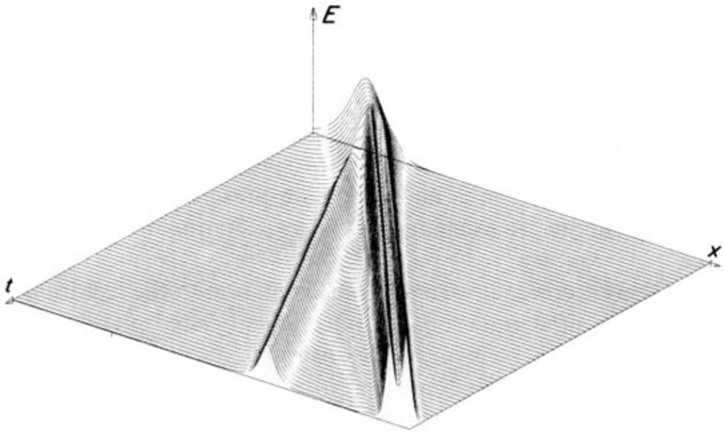

Figure 1. Calculated pulsed solutions of the triple SG. The initial condition is $\sigma = \sigma_1$. The initial pulse is a 6π hyperbolic secant. Break-up occurs with the double-peaked soliton of area $6\pi - 2\sigma_1$ leading.

this break-up. The initial state is $\sigma = 3\cos^{-1}[\frac{1}{6}(-1+\sqrt{7})] \equiv \sigma_1$. This is a zero of $\sin\sigma + \frac{2}{3}\sin\frac{2}{3}\sigma + \frac{1}{3}\sin\frac{1}{3}\sigma$ and is an unstable singular point of the triple SG. In the optical pulse problem $\sigma = \sigma_1$ represents, in the absence of spontaneous radiation processes, a *stable* equilibrium but excited dielectric of seven-fold degenerate two-level atoms or molecules. A second zero $6\pi - \sigma_1$ represents a similar equilibrium dielectric. It may be possible to create short-lived dielectrics of either type in molecular gases where infrared vibration–rotation transitions with $Q(J)$ symmetry have small Einstein A coefficients.

The break-up shown in figure 1 is unusual in two respects. Firstly it involves two distinct types of soliton distinguishable both by shape and area: the two-peaked soliton has area $6\pi - 2\sigma_1$, the one-peaked $2\sigma_1$, and neither is an integral multiple of 2π. The former takes a dielectric in the equilibrium state $\sigma = \sigma_1$ to one in the equilibrium state $6\pi - \sigma_1$. Both states have equal energy. The pulse of area $2\sigma_1$ takes the dielectric from state $\sigma = 6\pi - \sigma_1$ to the original state $\sigma = \sigma_1$. Soliton-like pulses must take dielectrics from equilibrium state to equilibrium state. Therefore with the initial condition $\sigma = \sigma_1$, break-up *must* take place with the two-peaked pulse leading. Similarly if $\sigma = 6\pi - \sigma_1$ initially, break-up must take place in reverse order. Figure 2 confirms this prediction: note the very broad double-peaked pulse at the back.

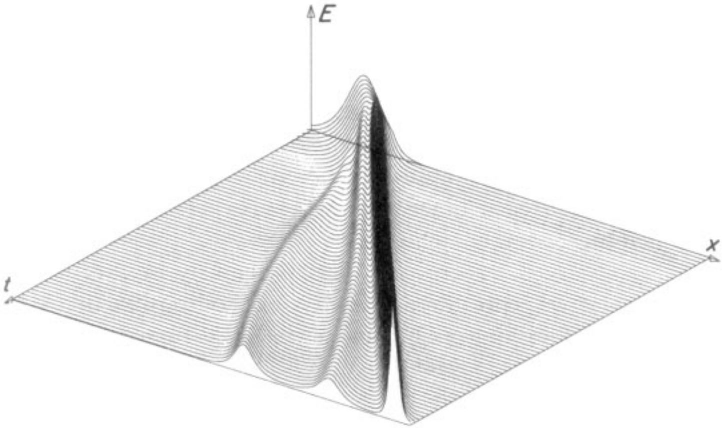

Figure 2. Calculated pulsed solutions of the triple SG when $\sigma = 6\pi - \sigma_1$ initially. A 6π hyperbolic secant now breaks with the single-peaked soliton of area $2\sigma_1$ leading.

It follows that in a collision between two such pulses in their proper orders the pulses maintain their order: area and shape are preserved in the collision but amplitude, velocity and energy are exchanged. This expectation also has been confirmed on the computer.

Equations (2) are Lorentz invariant: transformation to the rest frame yields second-order ordinary differential equations. From the associated phase planes we predict a variety of similar collision behaviours; for example, for $J = 4$ collisions between one one-peaked and one three-peaked pulse in proper order preserve that order. Such collisions contrast with the only previously known case of collision between dissimilar solitons (Caudrey *et al* 1973a). Here a 2π sech collides with a 0π solution and the two solitons emerge in reverse order after collision.†

Numerical methods cannot confirm the *exact* multisoliton character of the solutions of (2). But we see no evidence of deviations from exact soliton behaviour of the type reported by Kruskal (1974) for the Thirring equation $\sigma_{xx} - \sigma_{tt} = -\sigma + \pi^{-2}\sigma^3$. This equation approximates to the simple SG but does not have multisoliton solutions. Our present conclusion therefore is that equations (2) have a wealth of unusual solutions of multisoliton type.

This conclusion is a main point of this note. But a second point is to report that, despite this, equations (2) are apparently not soluble by the inverse scattering method as it is presently formulated. We have proved two theorems for nonlinear dispersive wave equations in $z(x, t)$ of the form

$$z_{xt} = F(z). \tag{3}$$

Equations (2) take this form after changing the independent variables to the characteristic lines $x \to x - t$; $t \to x + t$.

The theorems are as follows:

Theorem 1.
Two equations $z_{xt} = F(z)$, $z'_{xt} = G(z')$ have an invertible Bäcklund transformation (BT) $z \leftrightarrow z'$ if and only if F, G are solutions of $\ddot{F} - \alpha^2 F = 0$, $\ddot{G} - \alpha^2 h^{-2} G = 0$ for some

† Figure 1 in this paper shows break-up only. Details of the collision will be published.

complex-valued constant α, not excluding $\alpha = 0$, and for any real $h \neq 0$. If $\alpha \neq 0$ the BT is

$$z'_x = hz_x + \frac{2h}{\alpha k}\left[A \exp\left(\frac{\alpha}{2h}(z'-hz)\right) + hD \exp\left(-\frac{\alpha}{2h}(z'-hz)\right)\right]$$

$$z'_t = -hz_t + k\left[\exp\left(\frac{\alpha}{2h}(z'+hz)\right) - hCA^{-1} \exp\left(\frac{\alpha}{2h}(z'+hz)\right)\right]$$

between

$$z_{xt} = C\exp(\alpha z) + D\exp(-\alpha z)$$

$$z'_{xt} = A\exp\left(\frac{\alpha}{h}z\right) + B\exp\left(-\frac{\alpha}{h}z\right)$$

where $h^2CD = AB$ (we assume A, B, C, D so chosen that both z_{xt} and z'_{xt} are real). If $\alpha = \sqrt{-1}$, $h = 1$ and $A = C = -B = -D = \frac{1}{2}$, this BT is the well known auto-BT transforming between two solutions of the simple SG (Lamb 1971). McLaughlin and Scott (1973) have reported a form of theorem 1 for auto-BT.

Theorem 2.
The equation $z_{xt} = F(z)$ has an infinity of polynomial conserved densities (PCD) of every even rank if and only if

$$F(z) = A\exp(\alpha z) + B\exp(-\alpha z)$$

where α is some nonzero complex-valued constant.

Kruskal (1974) has stated a theorem included by ours. In essence Kruskal's theorem is that $z_{xt} = F(z)$ has the PCD $\frac{1}{2}z_x^2$ and has a further PCD if and only if $\ddot{F} - \alpha^2 F = 0$.

Chen (1974) shows that apparently† all nonlinear dispersive wave equations soluble by the inverse scattering method in the forms due to Zakharov and Shabat (1971) or Ablowitz *et al* (1973b) have auto-BT. We have shown (Dodd and Bullough 1975) by the methods of the former paper that all equations soluble by the inverse method have an infinity of PCD. Theorems 1 and 2 show that equations (2) have neither auto-BT nor an infinity of PCD and we actually find one PCD only in agreement with Rund (1974, private communication).

It follows that there are nonlinear dispersive wave equations (namely (2) with $J > 1$) which have multisoliton-type solutions which are not soluble by the inverse method. Caudrey and Gibbon (1975) have obtained analytical two- and multisoliton solutions of systems of equations which generalize and include the Boussinesq equation for which Hirota (1973) found a multisoliton solution. These equations do not appear to have auto-BT, an infinity of PCD, and a solution by the inverse method. Further work is needed on these several equations.

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† A rigorous proof is lacking.

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